On the Constant that Fixes the Area Spectrum in Canonical Quantum Gravity

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Abstract

The formula for the area eigenvalues that was obtained by many authors within the approach known as loop quantum gravity states that each edge of a spin network contributes an area proportional to $\sqrt{j(j+1)}$ times Planck length squared to any surface it transversely intersects. However, some confusion exists in the literature as to a value of the proportionality coefficient. The purpose of this rather technical note is to fix this coefficient. We present a calculation which shows that in a sector of quantum theory based on the connection $A = \Gamma - \gamma K$, where Γ is the spin connection compatible with the triad field, K is the extrinsic curvature and γ is Immirzi parameter, the value of the multiplicative factor is $8\pi\gamma$. In other words, each edge of a spin network contributes an area $8\pi\gamma\ell_P^2\sqrt{j(j+1)}$ to any surface it transversely intersects.

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One of the most intriguing results of canonical quantum gravity based on Ashtekar's new variables [1] and the loop representation of Rovelli and Smolin [2] is the discreteness of the area spectrum [3]. In this theory, there is Hilbert space of kinematical states with a basis given by spin networks, i.e., graphs in space with edges labelled by spins $j = \frac{1}{2}, 1, \ldots$ and vertices labelled by intertwining operators [4]. Each edge contributes an area proportional to $\sqrt{j(j+1)}$ times Planck length ℓ_P squared to any surface it transversely intersects. It was also realized by Immirzi [5] that an additional parameter enters the formula for the area spectrum due to a quantization ambiguity. Namely, instead of working with a complex connection $A_a = \Gamma_a - iK_a$, where Γ_a is the 3-dimensional spin connection compatible with the triad field and K_a is the extrinsic curvature of the spatial hypersurface, one can introduce real phase space variables of Barbero [6]:

$$
\begin{aligned}
{}^{\gamma}\!A_a &:= \Gamma_a - \gamma K_a \\
{}^{\gamma}\Sigma_{ab} &:= (1/\gamma)\Sigma_{ab},
\end{aligned}
\tag{1}
$$

where $\gamma > 0$ is real and is known as Immirzi parameter, and Σ_{ab} is the two-form dual to the densitized triad field. Then there is a one-parameter family of inequivalent quantizations depending on which of real SU(2) connections ${}^{\gamma}\!A_a$ have been used to construct the quantum theory (see Rovelli and Thiemann [5]). The parameter γ explicitly enters the formula for the area eigenvalues: each transverse intersection contributes an area proportional to $\gamma\ell_P^2\sqrt{j(j+1)}$ [5].

The purpose of this note is to find the value of the proportionality coefficient. To find this coefficient we start from the self-dual action [7] adjusting the multiplicative constant in front of it so that it coincides with Einstein-Hilbert action when the self-dual connection satisfies its equations of motion. One can then find Hamiltonian formulation of the theory, and perform a canonical transformation to the real variables (1). Knowing the simplectic structure on the phase space $({}^{\gamma}\!A, {}^{\gamma}\Sigma)$ one can use one of the methods [3] to calculate the area spectrum. Our result is that the proportionality constant is equal to 8π, i.e., the area spectrum is

$$8\pi\gamma\ell_P^2\sum_p\sqrt{j_p(j_p+1)}, \tag{2}$$

where the sum is taken over all points p on the surface where edges of a spin network state intersects this surface, j_p are spins (half-integers) that label the corresponding edges, γ is the real parameter as in (1), and $\ell_P^2 = G\hbar$, G being Newton constant.

We are aware of another attempt (see De Pietri and Rovelli [3]) to fix the value of the multiplicative factor in (2). However, Eq. (2.4) of their paper should read

$$\frac{1}{2G_{DR}}\int d^4x\sqrt{-g}R = \frac{1}{G_{DR}}\int dx^0\int d^3x\,[\cdots], \tag{3}$$

and as a consequence the constant $16\pi G_{DR}$ that the authors call Newton's constant is 2 times less the one usually called Newton's constant.

We start from the self-dual formulation of general relativity [7]. The action

$$\begin{aligned} S[\sigma, A] = -\frac{i}{8\pi G}\int_{\mathcal{M}} \mathrm{Tr}\,(\Sigma\wedge F) = \\ \frac{i}{8\pi G}\,\frac{1}{4}\int_{\mathcal{M}} d^4x\,\Sigma_{ab}^{AB}\,F_{cdAB}\tilde{\varepsilon}^{abcd} \end{aligned} \tag{4}$$

is a functional of tetrad $\sigma_a^{AA'}$ and the self-dual connection A_a^{AB} fields. Here $\mathcal{M}$ is the spacetime manifold (which we for simplicity assume to be of the topology $\mathbb{R}\times M$ where M is some compact manifold), primed and unprimed upper case letters stand for SL$(2, C)$ spinor indices, lower case letters denote spacetime indices, $\tilde{\varepsilon}^{abcd}$ is Levi-Civita density, and the tetrad field defines the metric via

$$g_{ab} = \sigma_a^{AA'}\sigma_{bAA'}. \tag{5}$$

The tetrad field σ is required to be anti-hermitian ($\overline{\sigma}_a^{AA'} = -\sigma_a^{A'A}$) so that the metric (5) is real Lorentzian metric of signature $(-+++)$. The self-dual connection A defines a derivative operator $\mathcal{D}_a$ that operates on unprimed spinors $\mathcal{D}_a\lambda_A = \partial_a\lambda_A + A_{aA}{}^B\lambda_B$. The field Σ in (4) is the two-form $\Sigma^{AB} := \sigma^{AA'}\wedge\sigma^B{}_{A'}$, or, in index notations $\Sigma_{ab}^{AB} = 2\sigma_{[a}^{AA'}\sigma_{b]}^B{}_{A'}$. Note that Σ is self-dual also in spatial indices

$$\frac{1}{2}\varepsilon^{abcd}\Sigma_{ab}^{AB} = i\,\Sigma^{cd\,AB}, \tag{6}$$

where ε is the natural volume 4-form defined by the metric. The field F in (4) is the curvature two-form of the connection A: $2\mathcal{D}_{[a}\mathcal{D}_{b]}\lambda_A = F_{abA}{}^B\lambda_B$.

We want to prove now that the action (4) is equal to the Einstein-Hilbert action

$$\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}R \tag{7}$$

when the connection A satisfies its equations of motion. Varying (4) with respect to A one gets (when σ is non-degenerate) $\mathcal{D}_a\sigma_b^{AA'} = 0$. This equation gives one a relation between curvature F and the Riemann curvature tensor: F turns out to be the self-dual part of the Riemann tensor (see [8], p. 292)

$$R_{abc}^d = F_{abB}^A\sigma_c^{BA'}\sigma_{AA'}^d + \overline{F}_{abB'}^{A'}\sigma_c^{AB'}\sigma_{AA'}^d. \tag{8}$$

Now, using self-duality (6) of Σ one can show that

$$F_{ab}^{AB} = -\frac{1}{4}{}^{+}R_{abcd}\Sigma^{cd\,AB} = -\frac{1}{4}R_{abcd}\Sigma^{cd\,AB}, \tag{9}$$

where ${}^{+}R_{abcd}$ is the self-dual part of Riemann tensor and self-duality of Σ was used to get the second identity.

Thus, when the connection satisfies its equations of motion the action (4) is equal to

$$\begin{aligned} &\frac{i}{8\pi G}\int_{\mathcal{M}} d^4x\, i\sqrt{-g}\,\Sigma^{cdAB}\frac{1}{2}F_{cdAB} = \\ &\frac{i}{8\pi G}\int_{\mathcal{M}} d^4x\, i\sqrt{-g}\,\Sigma^{cdAB}\left(-\frac{1}{8}\right)R_{cdef}\Sigma^{ef}_{AB}, \end{aligned} \tag{10}$$

where self-duality of Σ was used in the first line, and (9) in the second. Now, using identity [9]

$$\Sigma^{abAB}\Sigma^{cd}_{AB} = 4g^{a[c}g^{d]b} - 2i\epsilon^{abcd}, \tag{11}$$

we have for the action

$$\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R_{cdef}g^{a[c}g^{d]b} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R, \tag{12}$$

where the kinematical Bianchi identity $R_{[abc]}{}^d = 0$ was used to get the first formula. This proves that the action (4) is equivalent to Einstein-Hilbert action.

We now want to find a phase space formulation of the theory and perform a canonical transformation to the real variables (1). We use the covariant description in which the phase space is the space of solutions of the classical equations of motion. The simplectic structure on the phase space is determined by the 3-form $\mathbf{\Theta}$ that is found by varying the lagrangian $\mathbf{L}$ of the theory $\delta\mathbf{L} = \mathbf{E}\delta\phi + d\mathbf{\Theta}(\phi, \delta\phi)$, where ϕ denotes all of the dynamical fields and $\mathbf{E}$ is the 4-form giving the equations of motion (see, for example, [10]). For the case of theory defined by (4) we have

$$\mathbf{\Theta} = -\frac{i}{8\pi G}\mathrm{Tr}\,(\Sigma \wedge \delta A)\,. \tag{13}$$

Thus, the simplectic structure on the phase space is

$$\Omega|_{(A,\Sigma)}\,((\delta A, \delta\Sigma), (\delta A', \delta\Sigma')) = -\frac{i}{8\pi G}\int_M \mathrm{Tr}\,[\delta\Sigma \wedge \delta A' - \delta\Sigma' \wedge \delta A]\,, \tag{14}$$

where the integral is taken over a Cauchy surface M. As usual, it does not depend on a choice of M. The points of phase space are labelled by restrictions of fields A, Σ on M.

On M, A can be expressed in terms of real fields as $A = \Gamma - iK$, where Γ_a is the 3-dimensional spin connection compatible with the triad (the densitized triad vector field is dual to the two form Σ) and K is the extrinsic curvature of M. This suggests [5] that we introduce real phase space variables (1). Since M is compact, the canonical transformation to variables (1) is well-defined, and one can easily check that in terms of these manifestly real variables, the simplectic structure on the phase space is given by

$$\Omega|_{({}^\gamma A, {}^\gamma\Sigma)}\,((\delta^\gamma A, \delta^\gamma\Sigma), (\delta^\gamma A', \delta^\gamma\Sigma')) = \frac{1}{8\pi G}\int_M \mathrm{Tr}\,[\delta^\gamma\Sigma \wedge \delta^\gamma A' - \delta^\gamma\Sigma' \wedge \delta^\gamma A]\,. \tag{15}$$

To compare the simplectic structure found with the one that was the starting point in the calculation of the area spectrum in [3], we now convert all SU(2) indices to SO(3) indices via

$$\begin{aligned} {}^\gamma A_{aA}^{\ \ B} &:= -\frac{i}{2}\tau^{i\ B}_{A}\,{}^\gamma A^i_a \\ {}^\gamma \tilde{E}^{a\ B}_{A} &:= -\frac{i}{\sqrt{2}}\tau^{i\ B}_{A}\,{}^\gamma \tilde{E}^{ai}, \end{aligned} \tag{16}$$

which are the standard conventions in the literature (see, for example, [8]). Here τ_A^{iB} are Pauli matrices $(\tau^i\tau^j)_A^B = i\varepsilon^{ijk}\tau_A^{kB} + \delta^{ij}\delta_A^B$, and the densitized triad field ${}^\gamma\tilde{E}$ is related to ${}^\gamma\Sigma$ via

$$ {}^\gamma\Sigma_{ab}^{AB} = \sqrt{2}\underset{\sim}{\varepsilon}_{abc}\,{}^\gamma\tilde{E}_A^{cB}. \tag{17} $$

To get the last formula the relation [8], Appendix A (43') was used. Using these standard definitions and identities one can find the simplectic structure to be equal to

$$ \Omega|_{({}^\gamma A,{}^\gamma\tilde{E})}\left((\delta^\gamma A,\delta^\gamma\tilde{E}),(\delta^\gamma A',\delta^\gamma\tilde{E}')\right) = -\frac{1}{8\pi G}\int_M d^3x\left[\delta^\gamma\tilde{E}^{ai}\delta^\gamma A'_{ai} - \delta^\gamma\tilde{E}'^{ai}\delta^\gamma A_{ai}\right]. \tag{18} $$

Thus, the Poisson brackets between the canonical variables $({}^\gamma A, {}^\gamma\tilde{E})$ are given by

$$ \left\{{}^\gamma A_{ai}(x), {}^\gamma\tilde{E}^{bj}(y)\right\} = 8\pi G\delta_i^j\delta_a^b\tilde{\delta}(x,y). \tag{19} $$

The Poisson brackets in the form (19) is the starting point of the calculation of the area spectrum performed by Ashtekar and Lewandowski [3]. Comparing (19) with the formula (2.1) from that paper, we find $G_{AL} = 8\pi G$, where G_{AL} is the the constant of the dimension of Newton's constant used by Ashtekar and Lewandowski, and G is Newton's constant. We refer now to the calculation performed in that paper to fix the area spectrum to be (2).

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